# Technical Notes

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## Laminated Shell in Cylindrical Bending, Two-Dimensional Approach vs Exact

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### Introduction

THE results presented here compare through the thickness stress and displacement quantities generated from a two-dimensional laminated shell approach to exact elasticity solutions. The two-dimensional theory is that developed by the authors,[1,2] and includes a parabolic transverse shear strain distribution and a simplified large displacement and rotational geometric nonlinearity. For linear shell applications, the approach resembles that first proposed by Reddy.[3] However, the literature apparently does not include comparisons of the two-dimensional laminated shell formulation to the exact elasticity solutions due to Ren[4] and, hence, the versatility and limitations of the two-dimensional approach have not been fully explored. This technical note presents the comparisons.

### Laminated Shell Results

The theoretical approach and finite element casting of Refs. 1 and 2, (see Appendix for linear relationships) is applied to the cylindrical shell of Fig. 1. A $1 \times 20$ ($x$ by $s$) discretization for one-half of the shell length gives converged results.[5] The calculated displacements and stresses for 90 deg unidirectional laminates (fibers are parallel to the circumferential coordinate $s$) and [90/0/90] cross-ply laminates are nondimensionalized, as shown in Eq. (1), where $S = R/h$. The transverse displacement $w$ and the inplane circumferential stress, $\sigma_2$ are taken from the center of the laminate, i.e., at $s = L/2$. The transverse shear stress $\sigma_4$ and the circumferential displacement $u_2$ are taken from the left side of the laminate, i.e., at $s = 0$. The sign of the quantities of Eq. (1) is opposite that given by Ren[4] in some cases, since the respective coordinate systems are different. The changes are made so that physical and nondimensionalized quantities have the same sign.

$$\bar{w} = \frac{10E_2 w}{q_o h S^4}, \quad \bar{\sigma}_2 = \frac{\sigma_2}{-q_o S^2}, \quad \bar{\sigma}_4 = \left| \frac{\sigma_4}{q_o S} \right|, \quad \bar{v} = \frac{100 E_2 u_2}{q_o h S^3} \tag{1}$$

Tables 1–3 and Figs. 2–5 show both middle plane and through the thickness results. Table 1 shows that by allowing transverse shear degrees of freedom the shell response is much more flexible in the thicker laminates. Also, for the thick cases, the cross-ply deflections resulting from the present approach are less accurate than the unidirectional laminates, since equilibrium is not satisfied at the ply interfaces of the former, a direct consequence of two-dimensional assumptions.[1,2,5] Table 2 shows that the circumferential stress $\bar{\sigma}_2$ compares fairly well for both the unidirectional and cross-ply laminates. The present results are the most inaccurate when $R/h < 5$, and on the top surface of the shell, i.e., where the load is applied in the three-dimensional solution and the effects of the transverse normal stress (neglected in the present approach) are generally the most significant.[4] Two values are given for $\bar{\sigma}_4$ in Table 3; that indicated by an asterisk results from a refined $1 \times 20$ mesh, with smaller elements nearer the left support. This is necessary because in the thinner geometries the transverse shear stress varies very rapidly near the support. The accuracy of the transverse shear stress, $\bar{\sigma}_4$, for $S = 500$, as shown in Table 3, could be improved if a further mesh refinement were analyzed. Figures 3 and 4 show that the thicker the laminate, the more asymmetrical the exact transverse shear stress becomes; however, the present approach always gives a symmetric response about the middle plane. A cubic term in the stress expression for the present approach, which would give this asymmetry, was neglected as small.[2,5]

**Table 1 Transverse displacement, $\bar{w}$, for $s = L/2$; exact and classical solution (CST) taken from Ref. 4**

| | [90 deg] | | | [90 deg/0 deg/90 deg] | | |
|---|---|---|---|---|---|---|
| $R/h$ | $\bar{w}$ | Exact | CST | $\bar{w}$ | Exact | CST |
| 2 | 0.803 | 0.999 | 0.0764 | 1.141 | 1.436 | 0.0799 |
| 4 | 0.278 | 0.312 | 0.0752 | 0.382 | 0.457 | 0.0781 |
| 10 | 0.108 | 0.115 | 0.0749 | 0.128 | 0.144 | 0.0777 |
| 50 | 0.0762 | 0.0770 | 0.0748 | 0.0796 | 0.0808 | 0.0776 |
| 100 | 0.0751 | 0.0755 | 0.0748 | 0.0781 | 0.0787 | 0.0776 |
| 500 | 0.0746 | 0.0749 | 0.0748 | 0.0774 | 0.0773 | 0.0776 |

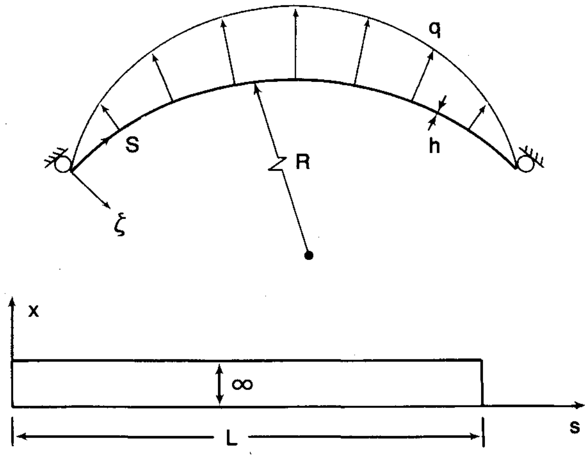


**Fig. 1 Cylindrical shell geometry, radius $R = 10$ in., arclength $L = 10.472$ in., thickness $h$. Shell is of infinite dimension parallel to the $\pm x$ direction; $E_1 = 25 \times 10^6$ psi, $E_2 = 1 \times 10^6$, $G_{12} = G_{13} = 0.5 \times 10^6$, $G_{23} = 2 \times 10^6$, $\nu_{12} = 0.25$. Loading, $q(x,s,0) = q \sin(\pi s/L)$.**

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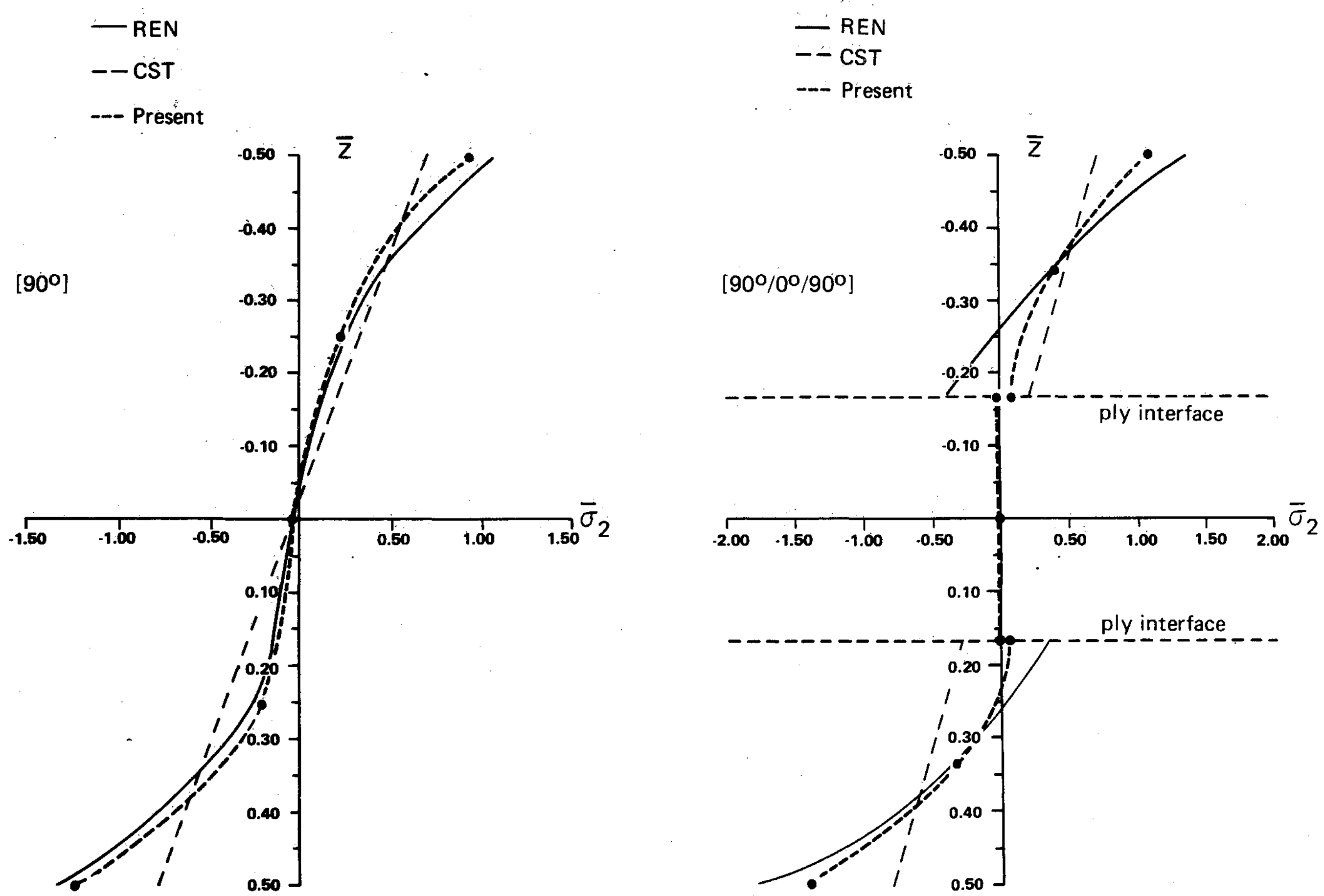


Fig. 2 Circumferential stress plotted as a function of thickness for $R/h = 4$. Ren and Classical Solution (CST) taken from Ref. 4, $\bar{Z} = \zeta/h$.

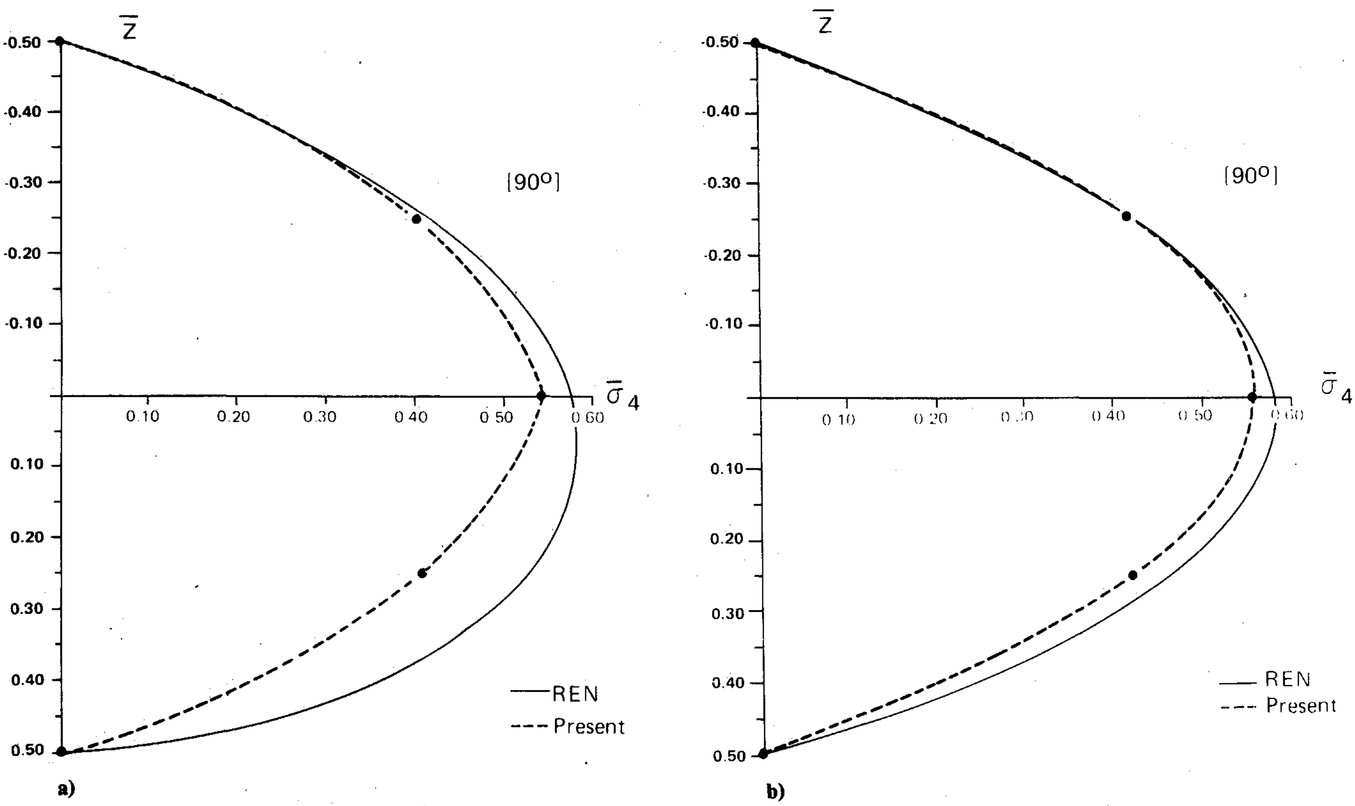


Fig. 3 Transverse shear stress plotted as a function of thickness for a) $R/h = 4$ and b) $R/h = 10$, unidirectional laminate, $\bar{Z} = \zeta/h$.

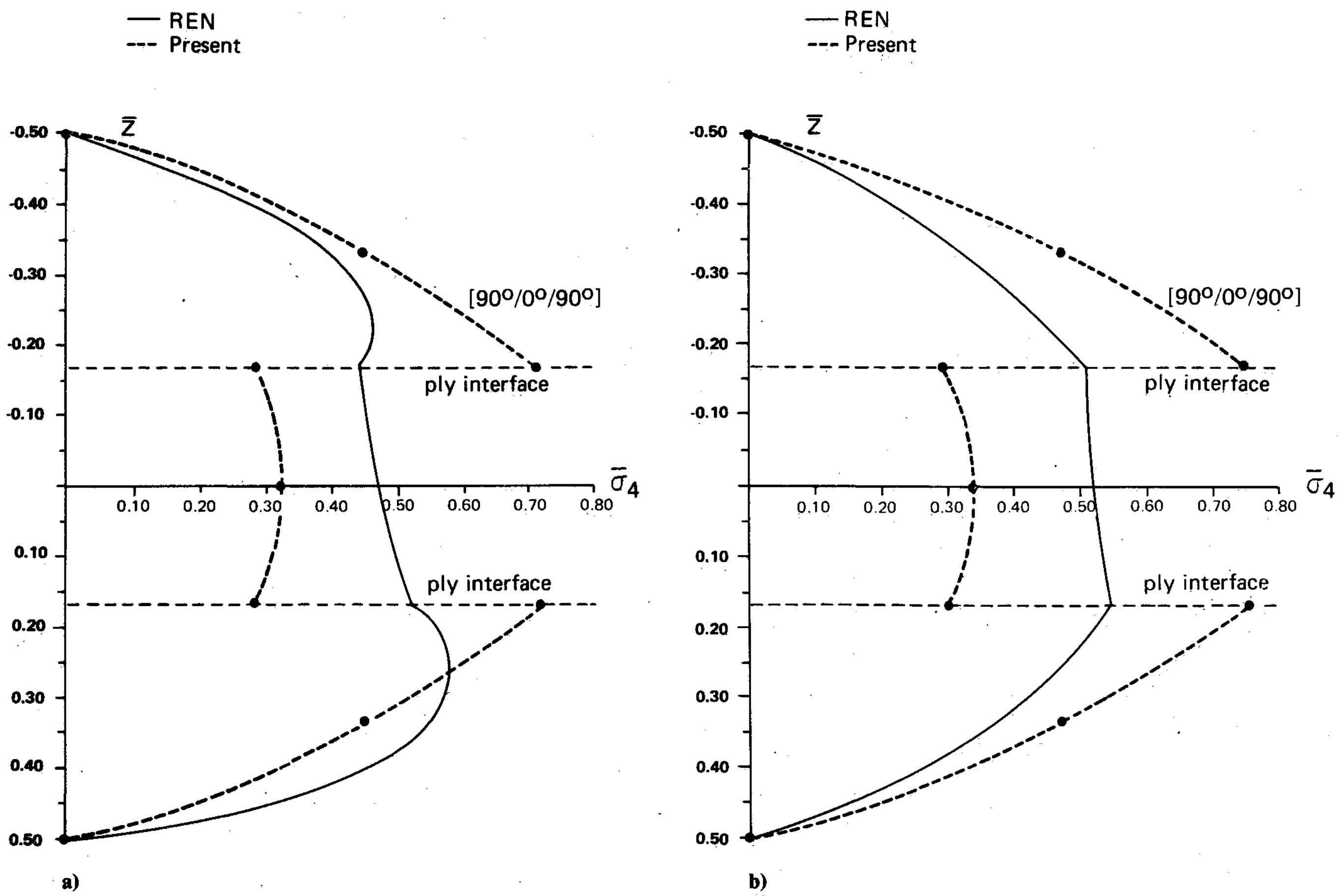


**Fig. 4 Transverse shear stress plotted as a function of thickness for a) $R/h = 4$ and b) $R/h = 10$, cross-ply laminate, $\bar{Z} = \zeta/h$.**

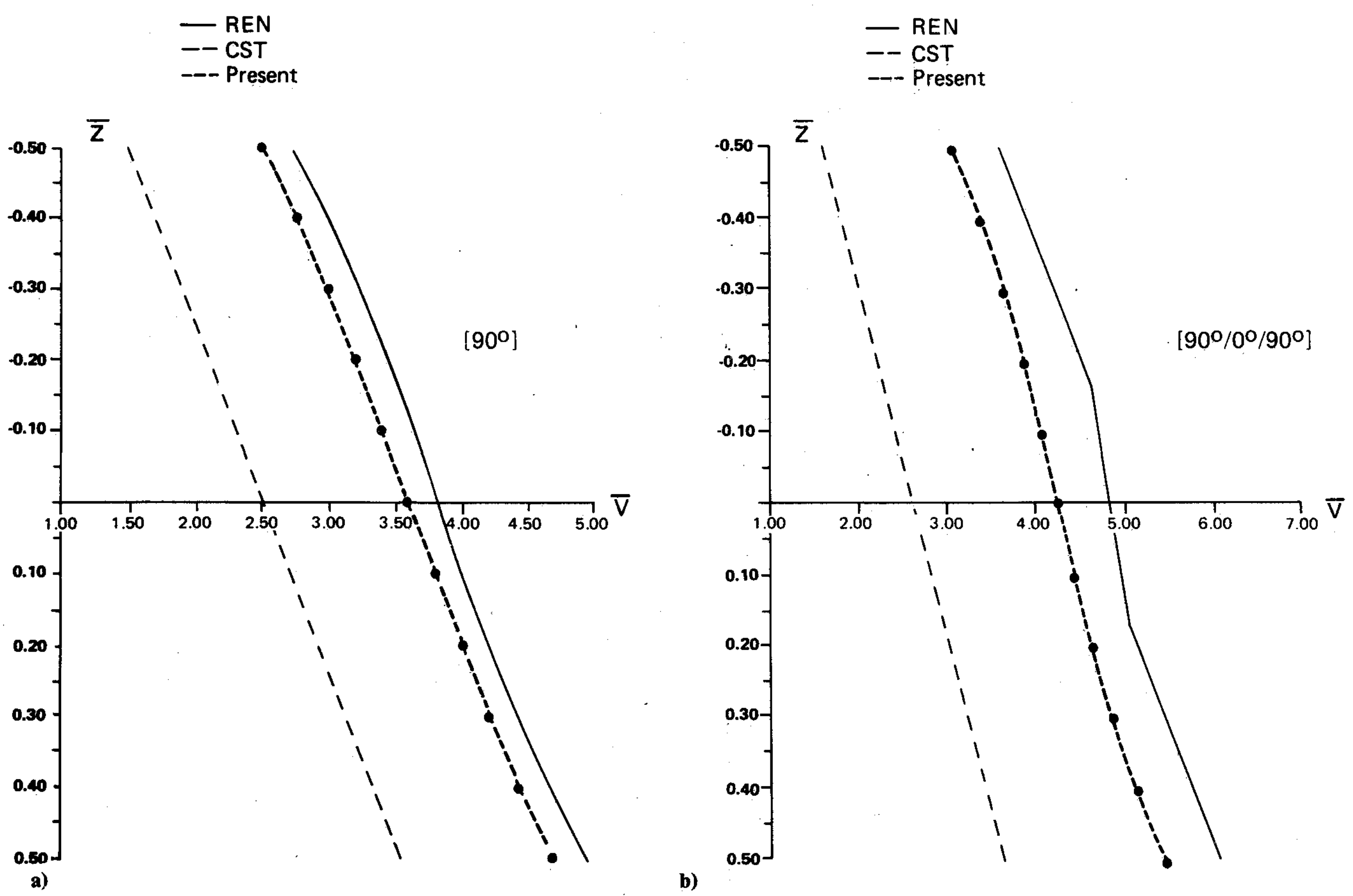


**Fig. 5 Circumferential displacement plotted as a function of thickness for $R/h = 4$. Ren and Classical Solution (CST) taken from Ref. 4, $\bar{Z} = \zeta/h$.**

**Table 2 Circumferential stress, $\bar{\sigma}_2$, for $s = L/2$, $\zeta = h/2$ (bottom) and $\zeta = -h/2$ (top); exact solution taken from Ref. 4**

| | [90 deg] | | | | [90 deg/0 deg/90 deg] | | | |
|---|---|---|---|---|---|---|---|---|
| | $\bar{\sigma}_2$ | | Exact | | $\bar{\sigma}_2$ | | Exact | |
| $R/h$ | $h/2$ | $-h/2$ | $h/2$ | $-h/2$ | $h/2$ | $-h/2$ | $h/2$ | $-h/2$ |
| 2 | −2.52 | 1.336 | −2.455 | 1.907 | −3.163 | 1.732 | −3.467 | 2.463 |
| 4 | −1.219 | 0.950 | −1.331 | 1.079 | −1.406 | 1.117 | −1.772 | 1.367 |
| 10 | −0.839 | 0.773 | −0.890 | 0.807 | −0.889 | 0.829 | −0.995 | 0.897 |
| 50 | −0.761 | 0.744 | −0.767 | 0.752 | −0.789 | 0.774 | −0.798 | 0.782 |
| 100 | −0.759 | 0.742 | −0.758 | 0.751 | −0.787 | 0.770 | −0.786 | 0.781 |
| 500 | −0.777 | 0.718 | −0.752 | 0.750 | −0.806 | 0.745 | −0.780 | 0.768 |

**Table 3 Transverse shear stress, $\bar{\sigma}_4$, for $s, \zeta = 0$; values with an asterisk result from refined 1×20 mesh; exact solution taken from Ref. 4**

| | [90 deg] | | [90 deg/0 deg/90 deg] | |
|---|---|---|---|---|
| $R/h$ | $\bar{\sigma}_4$ | Exact | $\bar{\sigma}_4$ | Exact |
| 2 | 0.489 | 0.555 | 0.287 | 0.394 |
| | 0.489* | | 0.287* | |
| 4 | 0.543 | 0.572 | 0.326 | 0.476 |
| | 0.543* | | 0.326* | |
| 10 | 0.558 | 0.579 | 0.339 | 0.525 |
| | 0.559* | | 0.340* | |
| 50 | 0.528 | 0.568 | 0.328 | 0.526 |
| | 0.557* | | 0.340* | |
| 100 | 0.449 | 0.565 | 0.291 | 0.523 |
| | 0.543* | | 0.334* | |
| 500 | 0.131 | 0.563 | 0.0853 | 0.525 |
| | 0.314* | | 0.217* | |

The figures show that this is a good assumption, since the asymmetry is appreciable only for the very thick $S = 4$ case. Figure 5 gives the circumferential displacement $\bar{V}$, taken at the left support for both laminate types.

## Conclusions

The results confirm that a relatively simple two-dimensional approach has a wide range of accuracy in predicting displacements and inplane stresses, and can be applied to more general laminates. However, for the very thick laminates [$(R/h < 5)$ for the cases studied], the three-dimensional effects of transverse normal stress and varying transverse displacement through the thickness play a major role in the shell response and, of course, the present two-dimensional approach cannot duplicate them.

## Appendix

Cylindrical shell kinematics:

$$u_1(x,s,\zeta) = u + \zeta\psi_1 + \zeta^3 k\,(\psi_1 + w_{,1})$$
$$u_2(x,s,\zeta) = v(1 - \zeta/R) + \zeta\psi_2 + \zeta^3 k(\psi_2 + w_{,2}) \tag{A1}$$
$$u_3(x,s) = w \qquad k = -4h^2/3$$

($u, v, w, \psi_1, \psi_2$ are functions only of $x$ and $s$, $(\ )_{,1}$ and $(\ )_{,2}$ are partial derivatives with respect to $x,s$ respectively).

Cylindrical shell strain-displacement (linear terms only):

$$\epsilon_1 = u_{,1} + \zeta\psi_{1,1} + \zeta^3 k(w_{,11} + \psi_{1,1})$$
$$\epsilon_2 = v_{,2} - w/R + \zeta\psi_{2,2}(1 + \zeta/R) + \zeta^3 k(w_{,22} + \psi_{2,2})(1 + \zeta/R)$$
$$\epsilon_6 = u_{,2}(1 + \zeta/R) + v_{,1}(1 - \zeta/R) + \zeta\psi_{1,2}(1 + \zeta/R) + \zeta\psi_{2,1}$$
$$+ \zeta^3 k w_{,12}(2 + \zeta/R) + \zeta^3 k\psi_{1,2}(1 + \zeta/R) + \zeta^4 k\psi_{2,1} \tag{A2}$$
$$\epsilon_4 = (1 + 3k\zeta^2)(w_{,2} + \psi_2), \quad \epsilon_5 = (1 + 3k\zeta^2)(w_{,1} + \psi_1)$$

The shell stresses are incorporated via the potential energy and are secondary functions, i.e., they are calculated using the displacements that are found from the finite element technique.

## References

[1]Dennis, S. T., and Palazotto, A. N., "Transverse Shear Deformation in Orthotropic Cylindrical Pressure Vessels Using a Higher Order Shear Theory," *AIAA Journal,* Vol. 27, No. 10, 1989, pp. 1441–1447.

[2]Dennis, S. T., and Palazotto, A. N., "Large Displacement and Rotational Formulation for Laminated Shells Including Parabolic Transverse Shear," *International Journal of Non-Linear Mechanics,* Vol. 25, No. 1, 1990, pp. 67–85.

[3]Reddy, J. N., and Liu, C. F., "A Higher Order Shear Deformation Theory of Laminated Elastic Shells," *International Journal of Engineering Science,* Vol. 23, No. 3, 1985, pp. 319–330.

[4]Ren, J. G., "Exact Solutions for Laminated Cylindrical Shells in Cylindrical Bending," *Composites Science and Technology,* Vol. 29, 1987, pp. 169–187.

[5]Dennis, S. T., *Two-Dimensional Laminated Shell Theory Including Parabolic Transverse Shear,* USAFA-TR-89-6, USAFA/DFEM, USAF Air Force Academy, Colorado Springs, CO, 1989.